AN EXPRESSION FOR THE SURFACE OF AN ELLIPSOID IN TERMS OF WEIERSTRASS'S ELLIPTIC FUNCTIONS.*

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It is proposed in this paper to give a simpler solution of a problem already solved by Weierstrass (Schwarz) and others.

The equation to the surface of an ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \,, \tag{1}$$

any point of it may be defined by the equations

$$\frac{x^2}{a^2} = \frac{(a^2 - u)(a^2 - v)}{(b^2 - a^2)(c^2 - a^2)}\,, \quad \frac{y^2}{b^2} = \frac{(b^2 - u)(b^2 - v)}{(a^2 - b^2)(c^2 - b^2)}\,, \quad \frac{z^2}{c^2} = \frac{(c^2 - u)(c^2 - v)}{(a^2 - c^2)(b^2 - c^2)}\,, \tag{2}$$

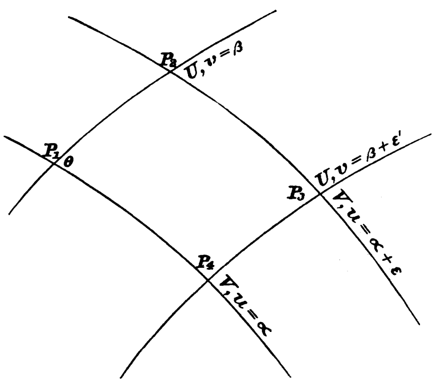
in which u and v are independent variables, called the "elliptic coordinates" of the point (x, y, z).†

Now the element of any surface defined by

$$x = f_1(u, v)\,, \quad y = f_2(u, v)\,, \quad z = f_3(u, v)\,, \tag{3}$$

in which f_1, f_2, f_3 are algebraic functions, may be expressed as follows:—

We suppose the surface defined by equations (3) has in it two systems of curves U and V. It is also supposed that the curves in each system U and V are consecutive; i. e. are indefinitely near to each other. Let there be two curves of the system U indefinitely near to each other, along each of which v is constant, viz. $v = \beta, \beta + \epsilon'$, respectively, where ϵ' is an infinitesimal of the first order; also, let there be two curves of the system V, indefinitely near to each other, for which u has the two constant values $u = \alpha, \alpha + \epsilon$, respectively. Let the point $(u = \alpha, v = \beta)$ be represented



* Read before the New York Mathematical Society, June 4, 1892.

† Joachimsthal's Anwendung der Differential- und Integral-rechnung auf die allgemeine Theorie der Flächen. § 68, p. 136.

by P_1, $(u = \alpha + \varepsilon,\ v = \beta)$ by P_2, $(u = \alpha + \varepsilon,\ v = \beta + \varepsilon')$ by P_3, $(u = \alpha,\ v = \beta + \varepsilon')$ by P_4. Then, since the curves are indefinitely near, the surface-element $P_1P_2P_3P_4$ may be regarded as a plane parallelogram (omitting infinitesimals of the second order), and its area will be $P_1P_2 \,.\, P_1P_4 \sin\theta$. If the coordinates of P_1 be x, y, z, those of P_2 are

$$x + \frac{\partial x}{\partial u}du, \quad y + \frac{\partial y}{\partial u}du, \quad z + \frac{\partial z}{\partial u}du\,;$$

and of P_4,

$$x + \frac{\partial x}{\partial v}dv, \quad y + \frac{\partial y}{\partial v}dv, \quad z + \frac{\partial z}{\partial v}dv\,.$$

Hence the values of $\cos\theta$, P_1P_2, P_1P_4, and the surface-element dL are

$$\cos\theta = \frac{\overline{P_1P_2}^2 + \overline{P_1P_4}^2 - \overline{P_2P_4}^2}{2P_1P_2 \,.\, P_1P_4} = \frac{E}{\sqrt{FG}},$$

$$P_1P_2 = \sqrt{F}\,.\,du,$$

$$P_1P_4 = \sqrt{G}\,.\,dv,$$

$$dL = \sqrt{FG - E^2}\,.\,du\,dv\,; \tag{4}$$

where

$$E = \frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v},$$

$$F = \left[\frac{\partial x}{\partial u}\right]^2 + \left[\frac{\partial y}{\partial u}\right]^2 + \left[\frac{\partial z}{\partial u}\right]^2,$$

$$G = \left[\frac{\partial x}{\partial v}\right]^2 + \left[\frac{\partial y}{\partial v}\right]^2 + \left[\frac{\partial z}{\partial v}\right]^2.$$

Since dL is the surface-element for the surface defined by equations (3), we may employ the expression for dL in (4) in the case of the ellipsoid by taking, instead of f_1, f_2, f_3, the values of x, y, z obtained from equations (2); then construct the *partial differential expressions* E, F, G; and finally $\sqrt{FG - E^2}$; whence

$$L = \int\!\!\int \sqrt{FG - E^2}\,.\,du\,.\,dv$$

$$= \tfrac{1}{4}\int\!\!\int \frac{(u - v)\,.\,u\,.\,v\,.\,du\,dv}{\sqrt{-(u - a^2)(u - b^2)(u - c^2)u}\,\sqrt{(v - a^2)(v - b^2)(v - c^2)v}}, \tag{5}$$

in which compatible with (1) and (2), there always exist the relations $a^2 > u > b^2 > v > c^2$.

We shall now show that (5) can be expressed in terms of σ- and p-functions.

Let

$$\begin{aligned}\varphi(t) &= (t-a^2)(t-b^2)(t-c^2)\,t \\ &= At^4 + 4Bt^3 + 6Ct^2 + 4Dt + R,\end{aligned} \tag{6}$$

where

$$C = \tfrac{1}{6}(b^2c^2 + c^2a^2 + a^2b^2), \quad D = -\tfrac{1}{4}a^2b^2c^2, \quad R = 0;$$

then taking a new variable s such that

$$s = \tfrac{1}{2}C + \frac{D}{t}, \quad \text{or} \quad t = \frac{D}{s-s_0},$$

where $s_0 = \frac{1}{2}C$, one may easily verify

$$\frac{dt}{\sqrt{\pm\varphi(t)}} = \frac{ds}{\sqrt{\pm S}}, \tag{7}$$

where

$$S = 4(s-e_1)(s-e_2)(s-e_3), \quad \frac{dt}{ds} > 0,$$

and

$$\begin{aligned} e_1 &= \tfrac{1}{2}C + \frac{D}{a^2} = \tfrac{1}{12}(a^2b^2 - 2b^2c^2 + c^2a^2), \\ e_2 &= \tfrac{1}{2}C + \frac{D}{b^2} = \tfrac{1}{12}(b^2c^2 - 2c^2a^2 + a^2b^2), \\ e_3 &= \tfrac{1}{2}C + \frac{D}{c^2} = \tfrac{1}{12}(c^2a^2 - 2a^2b^2 + b^2c^2), \\ e_1 &+ e_2 + e_3 = 0. \end{aligned} \tag{8}$$

Therefore, multiplying (7) separately by $t = \dfrac{D}{s-s_0}$ and $t^2 = \dfrac{D^2}{(s-s_0)^2}$, and integrating between the limits b^2 and t, which correspond to the limits e_2 and s, we obtain the transformations

$$\int_{b^2}^{t} \frac{t\,dt}{\sqrt{\pm\varphi(t)}} = \int_{e_2}^{s} \frac{D}{s-s_0} \cdot \frac{ds}{\sqrt{\pm S}}, \tag{9}$$

$$\int_{b^2}^{t} \frac{t^2 dt}{\sqrt{\pm\varphi(t)}} = \int_{e_2}^{s} \frac{D^2}{(s-s_0)^2} \frac{ds}{\sqrt{\pm S}};$$

and if we suppose s to take the values s_1, s_2, when t takes the values u, v, respectively, as in (5), then

$$s_1 = s_0 + \frac{D}{u}, \quad s_2 = s_0 + \frac{D}{v};$$

and from the inequalities $c^2 < v < b^2 < u < a^2$, with the fact that D is negative in (8), we have the conditions

$$e_3 < s_2 < e_2 < s_1 < e_1 < s_0 < \infty\ . \tag{11}$$

Also, placing

$$\begin{aligned} V_1 &= \int_{e_2}^{s_1} \frac{D}{s - s_0} \frac{ds}{\sqrt{-S}}, & V_2 &= \int_{s_2}^{e_2} \frac{D}{s - s_0} \frac{ds}{\sqrt{S}}, \\ V_3 &= \int_{e_2}^{s_1} \frac{D^2}{(s - s_0)^2} \frac{ds}{\sqrt{-S}}, & V_4 &= \int_{s_2}^{e_2} \frac{D^2}{(s - s_0)^2} \frac{ds}{\sqrt{S}}; \end{aligned} \tag{12}$$

there exist the relations

$$V_3 = D \frac{\partial V_1}{\partial s_0}, \qquad V_4 = D \frac{\partial V_2}{\partial s_0}; \tag{13}$$

hence (5) with the aid of (9), (10), (12), (13) can be put in the form

$$\psi (s_0, s_1, s_2) = V_3 V_2 - V_1 V_4 = D \left[\frac{\partial V_1}{\partial s_0} \cdot V_2 - V_1 \frac{\partial V_2}{\partial s_0} \right], \tag{14}$$

where $\psi (s_0, s_1, s_2)$ represents the half of the ellipsoid above the xy-plane.

We shall now use the Weierstrass definition of the p-function :—*

If

$$u = \int_z^\infty \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}} = \int_z^\infty \frac{ds}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}},$$

in which $e_1 > e_2 > e_3$ and $e_1 + e_2 + e_3 = 0$, then $z = \mathrm{p}\, u$.

Hence, if we put $i = \sqrt{-1}$ and write †

$$\begin{aligned} w &= {}'\!\!\int_{s_0}^{\infty} \frac{ds}{\sqrt{S}} = {}'\!\!\int_{-\infty}^{0} \frac{dt}{\sqrt{\varphi(t)}}, \\ \omega_2 &= {}'\!\!\int_{e_2}^{\infty} \frac{ds}{\sqrt{S}}, \\ k_1 &= {}'\!\!\int_{e_2}^{s_1} \frac{ds}{\sqrt{-S}} = {}'\!\!\int_{b^2}^{u} \frac{dt}{\sqrt{-\varphi(t)}}, \\ k_2 &= {}'\!\!\int_{s_2}^{e_2} \frac{ds}{\sqrt{S}} = {}'\!\!\int_{v}^{b^2} \frac{dt}{\sqrt{\varphi(t)}}; \end{aligned} \tag{15}$$

* See Enneper's Elliptische Functionen, § 6; or Halphen's Fonctions Elliptiques Ch. III, (1), (2).

† An accent written before a sign of integration indicates that the path of integration on the complex-plane is a straight line.

we shall have

$$\omega_2 - k_1 i = '\int_{s_1}^{\infty} \frac{ds}{\sqrt{S}},$$

$$\omega_2 + k_2 = '\int_{s_2}^{\infty} \frac{ds}{\sqrt{S}};$$

and

$$\left.\begin{array}{ll} s_0 = \mathrm{p}\, w, & e_2 = \mathrm{p}\, \omega_2\,, \\ s_1 = \mathrm{p}\,(\omega_2 - k_1 i)\,, & s_2 = \mathrm{p}\,(\omega_2 + k_2)\,. \end{array}\right\} \tag{16}$$

Using the relation $s_0 = \mathrm{p}\, w$ in (14) we obtain

$$\psi\,(s_0, s_1, s_2) = \frac{D}{\mathrm{p}'w}\left[\frac{\partial V_1}{\partial w} V_2 - V_1 \frac{\partial V_2}{\partial w}\right]. \tag{17}$$

We next transform the integrals in (12) to an independent variable k, where

$$s = \mathrm{p}\,(\omega_2 + k)\,, \quad \omega_2 + k = '\int_{s}^{\infty} \frac{ds}{\sqrt{S}},$$

$$dk = \frac{-\,ds}{\sqrt{S}} = \frac{-\,i ds}{\sqrt{-S}};$$

and, by (16), the limits e_2 and s_1 are to be replaced by 0, $-k_1 i$, and the limits e_2, s_2 by 0, k_2; whence

$$V_1 i = '\int_0^{-k_1 i} \frac{D dk}{\mathrm{p}\,(\omega_2 + k) - \mathrm{p}\, w}, \qquad V_2 = '\int_0^{k_2} \frac{D dk}{\mathrm{p}\,(\omega_2 + k) - \mathrm{p}\, w}; \tag{18}$$

and, multiplying by $\dfrac{\mathrm{p}'w}{D}$,

$$\left.\begin{array}{l} \dfrac{\mathrm{p}'w}{D} i\, V_1 = '\displaystyle\int_0^{-k_1 i} \frac{\mathrm{p}'w dk}{\mathrm{p}\,(\omega_2 + k) - \mathrm{p} w}, = i I_1 \quad \text{say}\,, \\ \dfrac{\mathrm{p}'w}{D}\, V_2 = '\displaystyle\int_0^{k_2} \frac{\mathrm{p}'w dk}{\mathrm{p}\,(\omega_2 + k) - \mathrm{p} w}, = I_2 \quad \text{``}\ . \end{array}\right. \tag{19}$$

We next calculate iI_1, I_2, by means of the formula*

$$\int_0^k \frac{\wp' w dk}{\wp(\omega_2+k)-\wp w} = \log\frac{\sigma(w-\omega_2)}{\sigma(w+\omega_2)} - \log\frac{\sigma(w-\omega_2+k)}{\sigma(w+\omega_2-k)} + 2\frac{\sigma'}{\sigma}(w)\,.\,k\,,$$

$$= \log\frac{\sigma(w-\omega_2)\,.\,\sigma(w-k+\omega_2)}{\sigma(w+k-\omega_2)\,.\,\sigma(w+\omega_2)} + 2\frac{\sigma'}{\sigma}(w)\,.\,k\,.$$

Also, we have the transformation formulæ †

$$\sigma(w-\omega_2) = -\sigma_2 w\,.\,\sigma\omega_2\,.\,e^{-\eta_2 . w},$$

$$\sigma(w+\omega_2) = \sigma_2 w\,.\,\sigma\omega_2\,.\,e^{\eta_2 . w},$$

$$\sigma(w-k+\omega_2) = \sigma_2(w-k)\,.\,\sigma\omega_2\,.\,e^{\eta_2(w-k)},$$

$$\sigma(w+k-\omega_2) = -\sigma_2(w+k)\,.\,\sigma\omega_2\,.\,e^{-\eta_2(w+k)}.$$

Finally, after substituting these values in the expression just deduced for $\int_0^k$ we shall have

$$\int_0^k \frac{\wp' w\,.\,dk}{\wp(\omega_2+k)-\wp w} = \log\frac{\sigma_2(w-k)}{\sigma_2(w+k)} + 2\frac{\sigma'}{\sigma}(w)\,.\,k\,. \tag{20}$$

For the integrals iI_1 and I_2 of (19) the limits are respectively $0, -k_1 i$ and $0, k_2$; whence, introducing these in (20), we obtain for the integrals of (19)

$$iI_1 = \log\frac{\sigma_2(w+k_1 i)}{\sigma_2(w-k_1 i)} - 2\frac{\sigma'}{\sigma}(w)\,.\,k_1 i\,,$$

$$I_2 = \log\frac{\sigma_2(w-k_2)}{\sigma_2(w+k_2)} + 2\frac{\sigma'}{\sigma}(w)\,.\,k_2\,. \tag{21}$$

Using the notation of (19) in (17), we have

$$\psi(s_0, s_1, s_2) = \left[\frac{\partial I_1}{\partial w} I_2 - I_1\frac{\partial I_2}{\partial w}\right]\left[\frac{D}{\wp' w}\right]^3. \tag{22}$$

But from (15) and (6), (8)

$$\frac{dw}{ds_0} = \left[\frac{1}{\sqrt{S}}\right]_{s=s_0}, = \frac{1}{D};$$

* Weierstrass and Schwarz's Elliptic Function Formulæ, § 60, (1);
† § 18, (3).

and, from (16)

$$\frac{ds_0}{dw} = \mathrm{p}'w \; ;$$

whence $\mathrm{p}'w/D = 1$; and (22) becomes

$$\psi\,(s_0, s_1, s_2) = \frac{\partial I_1}{\partial w} I_2 - I_1 \frac{\partial I_2}{\partial w}. \tag{23}$$

From (21)

$$i\,\frac{\partial I_1}{\partial w} = \frac{\sigma_2'}{\sigma_2}(w + k_1 i) - \frac{\sigma_2'}{\sigma_2}(w - k_1 i) + 2\,\mathrm{p}w\,.\,k_1 i\,,$$

$$\frac{\partial I_1}{\partial w} = \frac{\sigma_2'}{\sigma_2}(w - k_2) - \frac{\sigma_2'}{\sigma_2}(w + k_2) - 2\,\mathrm{p}w\,.\,k_2\,. \tag{24}$$

Again in (21) writing ω_3 for $-k_1 i$, and ω_1 for k_2, and using Weierstrass's formula* connecting $\sigma_2(w + 2\,\omega_3)$ with $\sigma_2 w$, we obtain

$$\log\frac{\sigma_2(w - \omega_3)}{\sigma_2(w + \omega_3)} = \log\frac{\sigma_2(w - \omega_3)}{e^{2\eta_3 w}\sigma_2(w - \omega_3)} = -\,2\eta_3\,w + 2n\pi i, \quad n = 0\,;$$

therefore

$$iI_1 = -\,2\eta_3 w + 2n\pi i + 2\,\frac{\sigma'}{\sigma}(w)\,.\,\omega_3\,, \quad n = 0\,,$$

$$I_2 = -\,2\eta_1 w + 2\,\frac{\sigma'}{\sigma}(w)\,.\,\omega_1\,; \tag{25}$$

and

$$i\,\frac{\partial I_1}{\partial w} = -\,2\,(\eta_3 + \mathrm{p}w\,.\,\omega_3)\,, \quad \frac{\partial I_2}{\partial w} = -\,2\,(\eta_1 + \mathrm{p}w\,.\,\omega_1)\,; \tag{26}$$

whence from (23)

$$\psi\,(s_0, s_1, s_2) = \frac{4}{i}\begin{vmatrix} -\,\eta_1\,w + \frac{\sigma'}{\sigma}(w)\quad \omega_1\,, & -\,\eta_3\,w + \frac{\sigma'}{\sigma}(w)\,.\,\omega_3 \\ -\,\eta_1 - \mathrm{p}w\,.\,\omega_1\,, & -\,\eta_3 - \mathrm{p}w\,.\,\omega_3 \end{vmatrix}, \tag{27}$$

which, on subtracting w times the second row of the determinant from the first and factoring, becomes

$$\psi\,(s_0, s_1, s_2) = \frac{4}{i}\left[\frac{\sigma'}{\sigma}(w) + w\,\mathrm{p}w\right]\begin{vmatrix} \omega_1\,, & \omega_3 \\ -\,\eta_1 - \mathrm{p}w\,.\,\omega_1\,, & -\,\eta_3 - \mathrm{p}w\,.\,\omega_3 \end{vmatrix}, \tag{28}$$

* Weierstrass and Schwarz, p. 22, 3rd column.

But the determinant in (28) reduces to

$$\eta_1 \omega_3 - \omega_1 \eta_3, = \tfrac{1}{2} \pi i ;\,^*$$

hence the total surface is

$$2 \psi (s_0, s_1, s_2) = 4\pi \left[\frac{\sigma'}{\sigma} (w) + w\, \mathrm{p} w \right]. \tag{29}$$

The method of computing $w = \int_0^\infty \frac{ds}{\sqrt{S}}$ may be found in treatises on Elliptic Functions, and the value of $\frac{\sigma'}{\sigma}(w)$ is obtained by taking the logarithmic differential of

$$\sigma w = e^{\frac{1}{6}\left(\frac{w\pi}{2\omega}\right)^2} \cdot \frac{2\omega}{\pi} \sin \frac{w\pi}{2\omega}, ^\dagger \tag{30}$$

while $\mathrm{p}w$ is given by the formula

$$\mathrm{p} w = - \frac{d^2}{dw} \log \sigma w, \tag{31}$$

thus completing the theoretical discussion of the problem.

By way of example we may take the case of the sphere, $r = a = b = c$, and from (6) and (16)

$$w = \int_{-\infty}^{0} \frac{dt}{\sqrt{\varphi(t)}} = \int_{-\infty}^{0} \frac{dt}{(t - a^2) \sqrt{t (t - a^2)}} ; \tag{32}$$

which on placing $\sqrt{t (t - a^2)} = z - t$, becomes

$$w = \int_{-\infty}^{0} \frac{2dz}{(z - a^2)^2} = \frac{2}{a^2}. \tag{33}$$

But from (8), when $a = b = c$, then

$$e_1 = e_2 = e_3 = 0, \quad \text{and} \quad g_2 = g_3 = 0 ;\,^\ddagger$$

whence it follows that

$$\frac{\sigma'}{\sigma} (w) + w\, \mathrm{p} w = \frac{2}{w} = a_2, \,^\S \tag{34}$$

and $2 \psi (s_0, s_1, s_2) = 4 \pi a^2$, the surface of the sphere.

* W. and S., § 7, (5).
† W. and S., § 6 (1) and § 7.
‡ Enneper, § 6, (21).
§ W. and S., § 8, (3); § 9, (6).

If $c = b$, then from (15) we shall have

$$w = \int_{-\infty}^{0} \frac{dt}{(t - b^2)\sqrt{t^2 - a^2 t}}, \tag{35}$$

which, by putting $\sqrt{t^2 - a^2 t} = z - t$, becomes

$$w = \int_{-\infty}^{0} \frac{2dz}{a^2 b^2 - 2 b^2 z + z^2} = \left[\frac{2}{abe} \tan^{-1} \frac{z - b}{abe}\right]_{-\infty}^{0}, \tag{36}$$

$$w = -\frac{3\pi}{abe} + \frac{2}{abe} \tan^{-1} \frac{b}{ae};$$

and from (8)

$$e_1 = \tfrac{1}{6} a^2 b^2 e^2, \quad e_2 = e_3 = -\tfrac{1}{12} a^2 b^2 e^2. \tag{37}$$

Therefore

$$g_2 = 3 e_1^2, \quad g_3 = e_1^3, \quad \left[\frac{2\omega}{\pi}\right]^2 = \frac{2g_2}{9g_3} = \frac{4}{a^2 b^2 e^2}.* \tag{38}$$

Weierstrass and Schwarz § 10, (1) and (3), furnish the equations

$$\mathrm{p}w = \left[\frac{\pi}{2\omega}\right]^2 \frac{1}{\sin^2\left[\frac{w\pi}{2\omega}\right]} - \tfrac{1}{3}\left[\frac{\pi}{2\omega}\right]^2,$$

$$\frac{\sigma'}{\sigma}(w) = \frac{\pi}{2\omega} \cot \frac{w\pi}{2\omega} + \tfrac{1}{3}\left[\frac{\pi}{2\omega}\right]^2 w; \tag{39}$$

$$\frac{\sigma'}{\sigma}(w) + w\,\mathrm{p}w = \frac{\pi}{2\omega}\left[\frac{w\pi}{2\omega} \frac{1}{\sin^2\left[\frac{w\pi}{2\omega}\right]} + \cot \frac{w\pi}{2\omega}\right]. \tag{40}$$

Multiplying (36) by $\frac{\pi}{2\omega} = \frac{abe}{2}$, we obtain

$$\frac{w\pi}{2\omega} = -\frac{3\pi}{2} + \tan^{-1} \frac{b}{ae}, \tag{41}$$

$$= -\frac{3\pi}{2} + \sin^{-1} \sqrt{1 - e^2}; \tag{42}$$

* Halphen's "Fonctions Elliptiques" V. I. p. 27.

whence,

$$\sin \frac{w\pi}{2\omega} = + \cos \sin^{-1} \sqrt{1 - e^2} = e \,,$$

$$\cos \frac{w\pi}{2\omega} = \sqrt{1 - e^2} \,,$$

$$\frac{w\pi}{2\omega} = \sin^{-1} e \,,$$

$$\cot \frac{w\pi}{2\omega} = \frac{\sqrt{1 - e^2}}{e} = \frac{b}{ae} \,. \tag{43}$$

Substituting these values for $\sin \frac{w\pi}{2\omega}$, $\frac{w\pi}{2\omega}$, $\cot \frac{w\pi}{2\omega}$, from (43), and $\frac{\pi}{2\omega} = \frac{abe}{2}$, from (38), in (40), we shall have

$$\frac{\sigma'}{\sigma}(w) + w \,\mathrm{p}w = \frac{ab}{2e} \sin^{-1} e + \frac{b^2}{2} \,; \tag{44}$$

and finally,

$$4\pi \left[\frac{\sigma'}{\sigma}(w) + w \,\mathrm{p}w \right] = 2\pi b^2 + 2\pi \frac{ab}{e} \sin^{-1} e \,, \tag{45}$$

the total surface of a spheroid.*

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* See Williamson's Integral Calculus (1884), p. 258 (1).

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